Exercise 1

Use the *Laplace transform method* to solve the Volterra integral equations of the first kind:

$$x - \sin x = \int_0^x (x-t)u(t)\,dt$$

Solution

The Laplace transform of a function $f(x)$ is defined as

$$\mathcal{L}\{f(x)\} = F(s) = \int_0^\infty e^{-sx} f(x)\,dx.$$

According to the convolution theorem, the product of two Laplace transforms can be expressed as a transformed convolution integral.

$$F(s)G(s) = \mathcal{L}\left\{\int_0^x f(x-t)g(t)\,dt\right\}$$

Take the Laplace transform of both sides of the integral equation.

$$\mathcal{L}\{x - \sin x\} = \mathcal{L}\left\{\int_0^x (x-t)u(t)\,dt\right\}$$

Use the fact that the Laplace transform is linear on the left side and apply the convolution theorem on the right side.

$$\begin{aligned}\mathcal{L}\{x\} - \mathcal{L}\{\sin x\} &= \mathcal{L}\{x\}U(s) \\ \frac{1}{s^2} - \frac{1}{s^2+1} &= \frac{U(s)}{s^2}\end{aligned}$$

Solve for $U(s)$.

$$\begin{aligned}U(s) &= 1 - \frac{s^2}{s^2+1} \\ &= \frac{1}{s^2+1}\end{aligned}$$

Take the inverse Laplace transform of $U(s)$ to get the desired solution.

$$\begin{aligned}u(x) &= \mathcal{L}^{-1}\{U(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\ &= \sin x\end{aligned}$$